An Explicit Construction of Self-dual 2-forms in Eight Dimensions

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Abstract

The geometry of self-dual 2-forms in 2n dimensional spaces is studied. These 2-forms determine a $n^2 - n + 1$ dimensional manifold $\mathcal{S}_{2n}$. We prove that $\mathcal{S}_{2n}$ has only one-dimensional linear submanifolds for n odd. In eight dimensions the self-dual forms of Corrigan et al constitute a seven dimensional linear subspace of $\mathcal{S}_8$ among many other equally interesting linear subspaces.

1. Introduction

The concept of self-duality of a 2-form in four dimensions is generalised to any higher even dimensional space in our previous paper [1]. We recall here that self-duality can be defined as an eigenvalue criterion in the following way. (Here we adopt a different terminology, and use self-dual rather than strongly self-dual as it is used in Ref.[1]) Suppose F is a real 2-form in 2n dimensions, and let Ω be the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some local orthonormal basis. Then by a change of basis, Ω can be brought to the block-diagonal form

$$\begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & \lambda_n \\ & & & & & -\lambda_n & 0 \end{pmatrix}$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of Ω. The 2-form F is called self-dual or anti-self-dual provided the absolute values of the eigenvalues are all equal , that is

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n|.$$

To distinguish between the two cases, orientation must be taken into account. We define F to be self-dual, if Ω can be brought with respect to an orientation-preserving basis change to the above block-diagonal form such that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Similarly, we define F to be anti-self-dual, if Ω can be brought to the same form by an orientation-reversing basis change. It is not difficult to check that in dimension D=4, the above definition coincides with the usual definition of self-duality in the Hodge sense. We studied in our previous work self-dual 2-forms in D=8 and have shown that

i) Yang-Mills 2-forms satisfying a certain set of 21 linear equations, first derived by Corrigan, Devchand, Fairlie, Nuyts (CDFN) [2] by other means, are self-dual in the above sense.

ii) Each self-dual 2-form F, satisfying $*(F \wedge F) = F \wedge F$ [3] is self-dual in the above sense.

In this letter, starting from the self-duality condition on eigenvalues we obtain the CDFN self-dual 2-form. We also explain the construction of new families of self-dual 2-forms.

2. The Geometry of Self-dual 2-forms.

In this section we describe the geometrical structure of self-dual 2-forms in arbitrary even dimensions. In the following I denotes an identity matrix of appropriate dimension.

Definition 1. Let $\mathbf{A}_{2n}$ be the set of antisymmetric matrices in $2n$ dimensions. Then $\mathcal{S}_{2n} = \{A \in \mathbf{A}_{2n} \mid A^2 + \lambda^2 I = 0, \lambda \in \mathbf{R}, \lambda \neq 0\}$.

Note that if $A \in \mathcal{S}_{2n}$, and $A^2 = 0$, then $A = 0$, and if $A \in \mathcal{S}_{2n}$, then $\lambda A \in \mathcal{S}_{2n}$ for $\lambda \neq 0$.

Proposition 2. *The set $\mathcal{S}_{2n}$ is diffeomorphic to $(O(2n) \cap \mathbf{A}_{2n}) \times \mathbf{R}^+$.*

Proof. Let $A \in \mathcal{S}_{2n}$ with $A^2 + \lambda^2 I = 0$. Note that $\lambda^2 = -\frac{1}{2n} trA^2$. Define $\kappa = [-\frac{1}{2n} trA^2]^{1/2}$, $\tilde{A} = \frac{1}{\kappa} A$. Then, $\tilde{A}^2 + I = 0$, hence $\tilde{A}\tilde{A}^\dagger = I$. Consider the map $\varphi : \mathcal{S}_{2n} \to (O(2n) \cap \mathbf{A}_{2n}) \times \mathbf{R}^+$ defined by $\varphi(A) = (\tilde{A}, \kappa)$. The map φ is 1-1, onto and differentiable. Its inverse is given by $(B, \alpha) \to \alpha B$ is also differentiable, hence φ is a diffeomorphism. e.o.p.

Remark 3. $O(2n) \cap \mathbf{A}_{2n}$ is a fibre bundle over the sphere S^{2n-2} with fibre $O(2n-2) \cap \mathbf{A}_{2n-2}$. (See Steenrod, Ref.[4])

For our purposes the following description of $\mathcal{S}_{2n}$ is more useful.

Proposition 4. *$\mathcal{S}_{2n}$ is diffeomorphic to the homogeneous manifold $(O(2n) \times \mathbf{R}^+)/U(n) \times \{1\}$, and $dim\mathcal{S}_{2n} = n^2 - n + 1$.*

Proof. Let G be the product group $O(2n) \times \mathbf{R}^+$, where $\mathbf{R}^+$ is considered as a multiplicative group. G acts on $\mathcal{S}_{2n}$ by $(P, \alpha)\dot{A} = \alpha(P^t AP)$, where $P \in O(2n)$, $\alpha \in \mathbf{R}^+$, $A \in \mathcal{S}_{2n}$, and t indicates the transpose. Since all matrices in $\mathcal{S}_{2n}$ are conjugate to each other up to a multiplicative constant, this action is transitive, and actually any $A \in \mathcal{S}_{2n}$ can be written as $A = \lambda P^t JP$, where $\lambda = [-\frac{1}{2n} trA^2]^{1/2}$, with $P \in O(2n)$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It can be seen that the isotropy subgroup of G at J is $U(n)$ [5] and $G/U(n)$ is diffeomorphic to $\mathcal{S}_{2n}$ (Ref.[6] p.132, Thm.3.62) Then $dim\mathcal{S}_{2n} = dim(O(2n) \times \mathbf{R}^+/U(n))$ can be easily computed as $dim\mathcal{S} = dimO(2n) + 1 - dimU(n) = (2n^2 - n + 1) - n^2 = n^2 - n + 1$. e.o.p.

In particular, in eight dimensions, $\mathcal{S}_8$ is a 13 dimensional manifold.

As $O(2n)$ has two connected components ($SO(2n)$ and $O(2n) \setminus SO(2n)$), $U(n)$ is connected and $U(n) \subset SO(2n)$, $\mathcal{S}_{2n}$ has two connected components. One of them (that contains J) consists of the selfdual forms and the other of the anti-self-dual forms.

3. Linear submanifolds of $\mathcal{S}_8$

The defining equations of the set $\mathcal{S}_8$ are homogeneous quadratic polynomial equations for the components of the curvature 2-form. Thus they correspond to differential equations quadratic in the first derivative for the connection. Thus the study of their solutions, hence the study of the moduli space of self-dual connections is rather difficult. Therefore one might hope to restrict the notion of self-duality, to the linear submanifolds of $\mathcal{S}_8 \cup \{0\}$. But there are plenty of them (at least in $\mathcal{S}_8$) and there is no plausible reason to single out a specific one of them. In Ref.[1] we have shown that the 2-forms satisfying a set of 21 equations proposed by Corrigan et al belong to $\mathcal{S}_8$ and we shall give below a natural way of arriving at them, but it will depend on a reference form. Changing the reference form one obtains translates of this 7-dimensional plane, which in some cases look more pregnant than the original one.

Note that we excluded the zero matrix from $\mathcal{S}_{2n}$ in our definition in order to obtain its manifold structure. We denote $\overline{\mathcal{S}}_{2n} = \mathcal{S}_{2n} \cup \{0\}$. By linearity of the action of $O(2n)$ on $\mathcal{S}_{2n}$ we obtain the following

Lemma 4. *Let $\mathcal{L}$ be a linear submanifold of $\overline{\mathcal{S}}_{2n}$. Then $\mathcal{L}_P = P^t \mathcal{L} P$, $P \in O(2n)$ is also a linear submanifold of $\overline{\mathcal{S}}_{2n}$.*

Let $J_o = diag(\epsilon, \epsilon, \epsilon, \epsilon)$, where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that any $A \in \mathcal{S}_8$ is conjugate to J_o, hence any linear subset of $\overline{\mathcal{S}}_8$ can be realized as the translate of a linear submanifold containing J_o under conjugation. Thus without loss of generality we can concentrate on linear subsets containing J_o.

Proposition 5. *If $A \in \mathcal{S}_8$ and $(A + J_o) \in \mathcal{S}_8$, where $J_o = diag(\epsilon, \epsilon, \epsilon, \epsilon)$, with $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then*

$$A = \begin{pmatrix} k\epsilon & r_1 S(\alpha) & r_2 S(\beta) & r_3 S(\gamma) \\ -r_1 S(\alpha) & k\epsilon & r_3 S(\gamma') & -r_2 S(\beta') \\ -r_2 S(\beta) & -r_3 S(\gamma') & k\epsilon & r_1 S(\alpha') \\ -r_3 S(\gamma) & r_2 S(\beta') & -r_1 S(\alpha') & k\epsilon \end{pmatrix}$$

where $k \in R$, r_1, r_2, r_3 are in R^+, and $S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, and α, α', β, β', γ, γ' satisfy

$$\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$$

.

Proof. If A and $A + J_o$ are both in $\mathcal{S}_8$, then the matrix $AJ_o + J_oA$ is proportional to identity. This gives a set of linear equations whose solutions can be obtained without difficulty to yield

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ -a_{12} & 0 & a_{14} & -a_{13} & a_{16} & -a_{15} & a_{18} & -a_{17} \\ -a_{13} & -a_{14} & 0 & a_{12} & a_{35} & a_{36} & a_{37} & a_{38} \\ -a_{14} & a_{13} & -a_{12} & 0 & a_{36} & -a_{35} & a_{38} & -a_{37} \\ -a_{15} & -a_{16} & -a_{35} & -a_{36} & 0 & a_{12} & a_{57} & a_{58} \\ -a_{16} & a_{15} & -a_{36} & a_{35} & -a_{12} & 0 & a_{58} & -a_{57} \\ -a_{17} & -a_{18} & -a_{37} & -a_{38} & -a_{57} & -a_{58} & 0 & a_{12} \\ -a_{18} & a_{17} & -a_{38} & a_{37} & -a_{58} & a_{57} & -a_{12} & 0 \end{pmatrix}$$

Then the requirement that the diagonal entries in A^2 be equal to each other give the following equations after some algebraic manipulations.

$$a_{13}^2 + a_{14}^2 = a_{57}^2 + a_{58}^2$$

$$a_{15}^2 + a_{16}^2 = a_{37}^2 + a_{38}^2$$

$$a_{17}^2 + a_{18}^2 = a_{35}^2 + a_{36}^2$$

Thus we can parametrize A by

$$a_{13} = r_1 \cos\alpha, \quad a_{15} = r_2 \cos\beta \quad a_{17} = r_3 \cos\gamma$$

$$a_{14} = r_1 \sin\alpha, \quad a_{16} = r_2 \sin\beta \quad a_{18} = r_3 \sin\gamma$$

$$a_{57} = r_1 \cos\alpha', \quad a_{37} = r_2 \cos\beta' \quad a_{35} = r_3 \cos\gamma'$$

$$a_{58} = r_1 \sin\alpha', \quad a_{38} = r_2 \sin\beta' \quad a_{36} = r_3 \sin\gamma'$$

Finally the requirement that the off diagonal terms in A^2 be equal to zero gives quadratic equations, which can be rearranged and using trigonometric identities they give $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$. e.o.p.

Thus the set of matrices $A \in \mathcal{S}_8$ such that $(A + J_o) \in \mathcal{S}_8$ constitutes an eight parameter family and the equations of CDFN correspond to the case $\alpha' + \alpha = \beta' + \beta = \gamma' + \gamma = 0$. Thus we have an invariant description of these equations, that we repeat here for convenience.

$$\begin{array}{lll} F_{12} - F_{34} = 0 & F_{12} - F_{56} = 0 & F_{12} - F_{78} = 0 \\ F_{13} + F_{24} = 0 & F_{13} - F_{57} = 0 & F_{13} + F_{68} = 0 \\ F_{14} - F_{23} = 0 & F_{14} + F_{67} = 0 & F_{14} + F_{58} = 0 \end{array}$$

$$\begin{array}{lll} F_{15}+F_{26}=0 & F_{15}+F_{37}=0 & F_{15}-F_{48}=0 \\ F_{16}-F_{25}=0 & F_{16}-F_{38}=0 & F_{16}-F_{47}=0 \\ F_{17}+F_{28}=0 & F_{17}-F_{35}=0 & F_{17}+F_{46}=0 \\ F_{18}-F_{27}=0 & F_{18}+F_{36}=0 & F_{18}+F_{45}=0 \end{array}$$

The (skew-symmetric) matrix of such a 2-form is

$$\begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} & F_{17} & F_{18} \\ & 0 & F_{14} & -F_{13} & F_{16} & -F_{15} & F_{18} & -F_{17} \\ & & 0 & F_{12} & F_{17} & -F_{18} & -F_{15} & F_{16} \\ & & & 0 & -F_{18} & -F_{17} & F_{16} & F_{15} \\ & & & & 0 & F_{12} & F_{13} & -F_{14} \\ & & & & & 0 & -F_{14} & -F_{13} \\ & & & & & & 0 & F_{12} \\ & & & & & & & 0 \end{pmatrix}$$

We will refer to the plane consisting of these forms as the CDFN-plane. Let us now consider as the reference form $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ instead of J_o. J can be obtained from J_o by conjugation $J = P^t J_o P$ with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the conjugation of the CDFN-plane by P is given by the following remarkable (D=8 self-dual) 2-form

$$F_{12}J + \begin{pmatrix} \Omega' & \Omega'' \\ -\Omega''^t & -\Omega' \end{pmatrix}$$

where Ω' is a D=4 self-dual 2-form and Ω'' is a D=4 anti-self-dual 2-form.

4. The Geometry of $\mathcal{S}_{4k+2}$

We prove that for odd n there are no linear subspaces other than the one dimensional one.

Proposition 6. *Let $\mathcal{M} = \{A \in \mathcal{S} \mid (A + J_o) \in \mathcal{S}\}$. Then $\mathcal{M} = \{kJ|k\epsilon\mathbf{R}\}$ for odd n.*

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^t & A_{22} \end{pmatrix}$, where $A_{11} + A_{11}^t = 0$, $A_{22} + A_{22}^t = 0$. As before if $(A + J_o) \in \mathcal{S}$ then $AJ_o + J_oA$ is proportional to the identity. This gives $A_{11} + A_{22} = 0$ and the symmetric part of A_{12} is proportional to identity. Therefore $A = kJ_o + \begin{pmatrix} A_{11} & A_{12o} \\ A_{12o} & -A_{11} \end{pmatrix}$, where A_{12o} denotes the antisymmetric part of A_{12} and k is a constant. Then the requirement that $A \in \mathcal{S}$ gives

$$[A_{11}, A_{12o}] = 0, \quad A_{11}^2 + A_{12o}^2 + kI = 0, \quad k \in R.$$

As A_{11} and A_{12o} commute, they can be simultaneously diagonalizable, hence for odd n they can be brought to the form

$$A_{11} = diag(\lambda_1\epsilon, \ldots, \lambda_{(n-1)/2}\epsilon, 0)$$

$$A_{12o} = diag(\mu_1\epsilon, \ldots, \mu_{(n-1)/2}\epsilon, 0)$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and 0 denotes a 1×1 block, up the the permutation of the blocks. If the blocks occur as shown, clearly $A_{11}^2 + A_{12o}^2$ cannot be proportional to identity. It can also be seen that except for $\lambda_i = \mu_i = 0$ the same result holds for any permutation of the blocks.

5. Conclusion

To conclude we would like to emphasise that the choice of a linear subspace of $\mathcal{S}_{2n}$ is incidental. Instead, one should try to understand the totality of the non-linear space of self-dual 2-forms. In that respect the approach to self-duality given above might be a good starting point.

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